Minisuperspace Models in M-theory

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Abstract

We derive the full canonical formulation of the bosonic sector of 11-dimensional supergravity, and explicitly present the constraint algebra. We then compactify M-theory on a warped product of homogeneous spaces of constant curvature, and construct a minisuperspace of scale factors. Classical and quantum behaviour of the minisuperspace system is then analysed, and quantum transition probabilities between classically disconnected regions of phase space are calculated. This behaviour turns out to be very similar to the "pre-Big Bang" scenario in quantum string cosmology.

1 Introduction

One of the most fundamental problems in theoretical physics is the search for a quantum theory which would unify gravity with other interactions. Over the past 20 years, superstring theory emerged as a successful candidate for this role. It was later discovered that all five superstring theories can all be obtained as special limits of a more general eleven dimensional theory known as M-theory and moreover, the low energy limit of which is the eleven dimensional supergravity [1, 2]. The complete formulation of M-theory is however not known yet.

In a cosmological context, there is another approach to quantum gravity which was pioneered by DeWitt in [3]. Here, the gravitational action is reformulated as a constrained Hamiltonian system and then quantized canonically. The resulting wavefunction is sometimes referred to as the "wavefunction of the universe" [4], as it describes the state of the universe. Such a wavefunction is a function on the superspace - an infinite dimensional space of all possible metrics modulo the diffeomorphisms. Although this procedure of course does not give a full theory of quantum gravity, it does give a low energy approximation, which is enough to capture some quantum effects such as tunnelling [4, 5]. Since the behaviour of the wavefunction in the full infinite-dimensional superspace is difficult to analyze, models with a reduced number of degrees of freedom have been considered. In these models only a finite subset of the original degrees of freedom are allowed to vary, while the rest are fixed, so that the wavefunction becomes a function on a finite-dimensional *minisuperspace*. In the early Universe, it is perceived that quantum gravity effects should become important, and so such minisuperspace models, where the degrees of freedom are the spatial scale factor in the Friedmann-Robertson-Walker (FRW) metric and possibly a scalar field, have been used to explore different quantum cosmological scenarios [4, 5, 6, 7, 8].

With the advent of superstring theory, the above ideas have been applied in the context of superstring theory [9, 10, 11, 12][13]. This time however, the starting point is the lowest order string effective action, possibly with a dilaton potential or a cosmological constant put in. Hence compared to the pure gravity case, there are new degrees of freedom - the dilaton and any of the tensor fields that appear. The minisuperspace models studied in the quantum string cosmology setting now have the FRW scale factor and the dilaton field as the independent degrees of freedom. In particular, progress has been made in quantum string cosmological description of the "pre-Big Bang scenario" [10, 11, 12, 14]. In this scenario, the universe evolves from a weakly coupled string vacuum state to a FRW geometry through a region of large curvature. Classically there is the problem that the pre-Big Bang and post-Big Bang branches are separated by a high-curvature singularity. However, in the minisuperspace model for a spatially flat ($k = 0$) FRW space-time with a suitable dilaton potential, it is possible to find a wavefunction which allows tunnelling between the two classically disconnected branches and this solves the problem of transition between the two regimes.

With M-theory being a good candidate to be a "theory of everything", it is interesting to see what the canonical quantization of the low energy effective theory can give, given the achievements of this approach in the pure gravity and superstring theory contexts. In Section 2 below, we start with the bosonic action for eleven-dimensional supergravity and reformulate the theory as a canonical constrained Hamiltonian system. The canonical formulation of eleven-dimensional supergravity has been considered before in [15] and [16], but here we explicitly give the constraint algebra, at least for the bosonic constraints. Then in Section 3, we reduce the system to a minisuperspace model. This is done by restricting the metric ansatz so that its spatial part is a warped product of a number of homogeneous spaces of constant curvature, and the supergravity 4-form is also restricted so that only its 4-space components are allowed to be non-zero. In the case when only one of the spatial components has non-vanishing curvature and the 4-form vanishes completely, it is possible to solve exactly both the classical equations of motion and the corresponding equation for the wavefunction. In section 4, we consider the classical and quantum solutions in the cases of vanishing, negative and positive spatial curvature. It turns out that the positive and negative curvature cases exhibit very similar behaviour to the string theory minisuperspace models described above with negative and positive dilaton potentials in the Hamiltonian respectively. Spatially flat M-theory minisuperspace models have also been considered in [17]. In section 5, we look at the case where the 4-form is switched on, the 3-space is flat, and one other spatial component is of positive curvature.

We will be using the following conventions. The spacetime signature will be taken as $(-++...+)$ and all the curvature conventions are the same as in [18]. Greek indices $\mu, \nu, \rho, ...$ range from 0 to 10, while the indices $\alpha, \beta, \gamma, ...$ range from 0 to 3. Latin indices $a, b, c, ...$ range from 1 to 10. The units used are such that $\hbar = c = 16\pi G^{(11)} = 1$.

2 Canonical formulation

In this section we set up the canonical formalism for the bosonic sector of 11-dimensional supergravity, with the field content being just the metric $\hat{g}_{\mu\nu}$ and the 3-form potential $\hat{A}$, with field strength $F = d\hat{A}$.

The action for the bosonic fields is [19]:

$$\begin{aligned} S &= \int d^{11}x\, (-\hat{g})^{\frac{1}{2}}\, \hat{R} - \frac{1}{2}\int F \wedge *F - \frac{1}{6}\int \hat{A} \wedge F \wedge F \\ &= \int d^{11}x\, (-\hat{g})^{\frac{1}{2}} \left(R^{(11)} - \frac{1}{48} F^{\mu_1 \cdots \mu_4} F_{\mu_1 \cdots \mu_4} - \frac{1}{12^4} \varepsilon^{\mu_1 \cdots \mu_{11}} \hat{A}_{\mu_1 \mu_2 \mu_3} F_{\mu_4 \cdots \mu_7} F_{\mu_8 \cdots \mu_{11}} \right) \end{aligned} \quad (1)$$

where $\hat{g} = \det[\hat{g}_{\mu\nu}]$ and $F_{\mu_1 \cdots \mu_4} = 4\partial_{[\mu_1} \hat{A}_{\mu_2 \mu_3 \mu_4]}$. The 11-dimensional alternating tensor $\varepsilon^{\mu_1 \cdots \mu_{11}}$

is defined by

$$\begin{aligned} \varepsilon^{\mu_1 \cdots \mu_{11}} &= (-\hat{g})^{-\frac{1}{2}} \, \eta^{\mu_1 \cdots \mu_{11}} \\ \varepsilon_{\mu_1 \dots \mu_{11}} &= (-\hat{g})^{\frac{1}{2}} \, \eta_{\mu_1 \dots \mu_{11}} \end{aligned}$$

where $\eta^{\mu_1 \cdots \mu_{11}} = -\eta_{\mu_1 \dots \mu_{11}}$ is the alternating symbol.

To decompose the metric into spatial and temporal parts, we use the following ansatz [3]:

$$\hat{g}_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_a \beta^a & \beta_a \\ \beta_b & \gamma_{ab} \end{pmatrix}. \tag{2}$$

The inverse metric is given by

$$\hat{g}^{\mu\nu} = \begin{pmatrix} -\alpha^{-2} & \alpha^{-2}\beta^a \\ \alpha^{-2}\beta^b & \gamma^{ab} - \alpha^{-2}\beta^a\beta^b \end{pmatrix} \tag{3}$$

where $\gamma_{ac}\gamma^{bc} = \delta_a^b$, and $\beta^a = \gamma^{ab}\beta_b$.

Using this ansatz, we follow [18] to express canonically the gravitational action.

Consider a hypersurface Σ, given by $t = const$. The future-pointing normal vector n^μ to this hypersurface is given by

$$n^\mu = \begin{pmatrix} \alpha^{-1} \\ -\alpha^{-1}\beta^a \end{pmatrix} \tag{4}$$

and the corresponding covector is $n_\mu = (-\alpha, \mathbf{0})$, so hence $n_\mu n^\mu = -1$.

The second fundamental form $K_{\mu\nu}$ for Σ is defined by

$$K_{\mu\nu} = -h_\mu^\rho h_\nu^\sigma n_{\rho;\sigma} \tag{5}$$

where the semicolon denotes covariant differentiation with respect to the 11-dimensional metric $\hat{g}$ and h_μ^ρ is the projector onto Σ defined by

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu.$$

Note that the sign in (5) depends on the convention used, so here we follow [18].

From (5) we have in particular

$$K_{ab} = -n_{a;b}. \tag{6}$$

Using the definition of n^μ (4), it can be shown that [18]

$$K_{ab} = \frac{1}{2}\alpha^{-1}\left(\beta_{a|b} + \beta_{b|a} - \gamma_{ab,0}\right) \tag{7}$$

where $|$ denotes covariant differentiation with respect to the metric γ_{ab}. Using the Gauss-Codazzi equation (8) below, the full eleven-dimensional curvature can be expressed in terms the intrinsic curvature of the hypersurface (that is, the curvature of the metric γ_{ab}) and the second fundamental form:

$$R^{(11)} = R^{(10)} - K_{ab}K^{ab} + K^2 - 2n^\mu n^\nu R_{\mu\nu}^{(11)} \tag{8}$$

where $K = \gamma^{ab}K_{ab}$ and $K^{ab} = \gamma^{ac}\gamma^{bd}K_{cd}$. Hence the gravitational Lagrangian density $\mathcal{L}_{grav}$ is given by

$$\begin{aligned} \mathcal{L}_{grav} &= (-\hat{g})^{\frac{1}{2}} \, R^{(11)} \\ &= \alpha\gamma^{\frac{1}{2}}\left(R^{(10)} - K_{ab}K^{ab} + K^2 - 2n^\mu n^\nu R_{\mu\nu}^{(11)}\right) \\ &= \alpha\gamma^{\frac{1}{2}}\left(R^{(10)} + K_{ab}K^{ab} - K^2\right) + \text{total derivative terms} \end{aligned} \tag{9}$$

where $\gamma = \det(\gamma_{ab})$. In the action, the full derivative terms give rise to a surface integral. We neglect it, since it does not affect the dynamics of the system.

We now decompose the 3-form $\hat{A}_{\mu\nu\rho}$ as

$$\hat{A}_{0ab} = B_{ab} \tag{10}$$
$$\hat{A}_{abc} = A_{abc} \tag{11}$$

and correspondingly,

$$F_{abcd} = 4\partial_{[a}A_{bcd]} \tag{12}$$
$$F_{0abc} = \partial_0 A_{abc} - 3\partial_{[a}B_{bc]}. \tag{13}$$

The F^2 term from the action (1) is decomposed as

$$F^{\mu_1\ldots\mu_4}F_{\mu_1\ldots\mu_4} = F_{abcd}F^{abcd} - 4F_{\perp bcd}F_{\perp}{}^{bcd} \tag{14}$$

where

$$F_{\perp bcd} = n^\mu F_{\mu bcd} = \alpha^{-1}\left(F_{0bcd} - \beta^a F_{abcd}\right). \tag{15}$$

Looking at the Chern-Simons term $A \wedge F \wedge F$, we have

$$\begin{aligned}\eta^{\mu_1\ldots\mu_{11}}\hat{A}_{\mu_1\mu_2\mu_3}F_{\mu_4\ldots\mu_7}F_{\mu_8\ldots\mu_{11}} = {}& \eta^{a_1\ldots a_{10}}\Big(12^2 B_{a_1a_2}\partial_{[a_3}A_{a_4a_5a_6]}\partial_{[a_7}A_{a_8a_9a_{10}]} \\ &+48\eta^{a_1\ldots a_{10}}\left(\partial_0 A_{a_1a_2a_3}\right)A_{a_4a_5a_6}F_{a_7a_8a_9a_{10}}\Big) + \text{total derivative term}\end{aligned} \tag{16}$$

Again, we neglect the total derivative term, since it does not affect the equations of motion.

Bringing together (9), (14) and (16), we thus have the total Lagrangian

$$L_{tot} = \int d^{10}x \left(\mathcal{L}_{grav} + \mathcal{L}_{form}\right)$$

where

$$\mathcal{L}_{grav} = \gamma^{\frac{1}{2}}\alpha\left(R^{(10)} + K_{ab}K^{ab} - K^2\right) \tag{17}$$

$$\begin{aligned}\mathcal{L}_{form} = {}& \gamma^{\frac{1}{2}}\Big[-\frac{\alpha}{48}F_{abcd}F^{abcd} + \frac{\alpha}{12}F_{\perp bcd}F_{\perp}{}^{bcd} \\ &-\varepsilon^{a_1\ldots a_{10}}\left(\frac{1}{12^2}B_{a_1a_2}\partial_{[a_3}A_{a_4a_5a_6]}\partial_{[a_7}A_{a_8a_9a_{10}]} - \frac{8}{12^4}\left(\partial_0 A_{a_1a_2a_3}\right)A_{a_4a_5a_6}F_{a_7a_8a_9a_{10}}\right)\Big]\end{aligned} \tag{18}$$

We see that the canonical fields in this system are $\alpha, \beta^a, \gamma^{ab}$ which come from the gravitational Lagrangian, together with A_{abc} and B_{ab} which come from $\mathcal{L}_{form}$. From the Lagrangian densities (17) and (18) we can now write down the canonical momenta conjugate to these variables:

$$\pi = \frac{\partial\mathcal{L}_{tot}}{\partial\alpha_{,0}} = 0 \tag{19a}$$
$$\pi^a = \frac{\partial\mathcal{L}_{tot}}{\partial\beta_{a,0}} = 0 \tag{19b}$$
$$p^{ab} = \frac{\partial\mathcal{L}_{tot}}{\partial B_{ab,0}} = 0 \tag{19c}$$
$$\pi^{ab} = \frac{\partial\mathcal{L}_{tot}}{\partial\gamma_{ab,0}} = -\gamma^{\frac{1}{2}}\left(K^{ab} - \gamma^{ab}K\right) \tag{19d}$$
$$\pi^{abc} = \frac{\partial L_{tot}}{\partial A_{abc,0}} = \frac{1}{6}\gamma^{\frac{1}{2}}F_{\perp}{}^{abc} - \frac{8}{12^4}\eta^{abcd_1\ldots d_7}A_{d_1d_2d_3}F_{d_4\ldots d_7}. \tag{19e}$$

Expressions (19a), (19b) and (19c) are known as *primary constraints* [20]. This means that the corresponding "velocities" cannot be expressed in terms of the momenta, and are thus arbitrary.

Now that we have the canonical momenta, we can work out the Hamiltonian for this system. The canonical Hamiltonian is given by

$$H_{tot} = \int d^{10}x \left(\alpha_{,0}\pi + \beta_{a,0}\pi^a + \gamma_{ab,0}\pi^{ab} + B_{ab,0}p^{ab} + A_{abc,0}\pi^{abc} - \mathcal{L}_{grav} - \mathcal{L}_{form} \right).$$

From [3], we know that the gravitational Hamiltonian H_{grav} is given by

$$H_{grav} = \int d^{10}x \left(\alpha_{,0}\pi + \beta_{a,0}\pi^a + \alpha\mathcal{H} + \beta_a\chi^a \right) \tag{20}$$

with

$$\begin{aligned} \mathcal{H} &= \gamma^{\frac{1}{2}} \left(K^{ab}K_{ab} - K^2 - R^{(10)} \right) = G_{abcd}\pi^{ab}\pi^{cd} - \gamma^{\frac{1}{2}} R^{(10)} && \text{(21a)} \\ \chi^a &= -2\pi^{ab}{}_{|b} = -2\pi^{ab}{}_{,b} - \gamma^{ad} \left(2\gamma_{bd,c} - \gamma_{bc,d} \right) \pi^{bc}. && \text{(21b)} \end{aligned}$$

where

$$G_{abcd} = \frac{1}{2}\gamma^{-\frac{1}{2}} \left(\gamma_{ac}\gamma_{bd} + \gamma_{ad}\gamma_{bc} - \frac{2}{9}\gamma_{ab}\gamma_{cd} \right) \tag{22}$$

is the Wheeler-DeWitt metric.

Consider the remaining part

$$H_{form} = \int d^{10}x \left(B_{ab,0}p^{ab} + A_{abc,0}\pi^{abc} - \mathcal{L}_{form} \right). \tag{23}$$

Due to the constraint (19c), nothing can be done with the first term, but in $\left(A_{abc,0}\pi^{abc} - \mathcal{L}_{form} \right)$ we have terms in $A_{abc,0}$, but these can be expressed in terms of π^{abc} using (19e). First, define

$$\tilde{\pi}^{abc} = \frac{1}{6}\gamma^{\frac{1}{2}} F_{\perp}{}^{abc} \tag{24}$$

so that, from (19e),

$$\tilde{\pi}^{abc} = \pi^{abc} + \frac{8}{12^4}\eta^{abcd_1\ldots d_7} A_{d_1d_2d_3} F_{d_4\ldots d_7}. \tag{25}$$

Then from definition of $F_{\perp abc}$ (15), we have

$$A_{bcd,0} = \beta^a F_{abcd} + 3\partial_{[b}B_{cd]} + 6\alpha\gamma^{-\frac{1}{2}}\tilde{\pi}_{bcd}. \tag{26}$$

Using (26) to substitute $A_{bcd,0}$ for π_{bcd}, we can write down the overall Hamiltonian in the form

$$H_{tot} = \int d^{10}x \left(\alpha_{,0}\pi + \beta_{a,0}\pi^a + B_{ab,0}p^{ab} + \alpha\tilde{\mathcal{H}} + \beta_a\tilde{\chi}^a + B_{ab}\tilde{\chi}^{ab} \right) \tag{27}$$

where

$$\begin{aligned} \tilde{\mathcal{H}} &= \mathcal{H} + \frac{1}{48}\gamma^{\frac{1}{2}} F_{abcd}F^{abcd} + 3\gamma^{-\frac{1}{2}}\tilde{\pi}^{abc}\tilde{\pi}_{abc} && \text{(28a)} \\ \tilde{\chi}^a &= \chi^a + F^a_{\ bcd}\tilde{\pi}^{bcd} && \text{(28b)} \\ \tilde{\chi}^{ab} &= -3\tilde{\pi}^{abc}{}_{,c} + \frac{1}{12^2}\eta^{aba_3\ldots a_{10}}\partial_{[a_3}A_{a_4a_5a_6]}\partial_{[a_7}A_{a_8a_9a_{10}]}. && \text{(28c)} \end{aligned}$$

We see from the Hamiltonian (27) that the quantities α, β_a and B_{ab} are arbitrary, so we set the gauge as convenient.

In order for the primary constraints (19a)-(19e) to be consistent with the equations of motion, the time derivatives of π, π^a and p^{ab} must vanish. This corresponds to vanishing Poisson brackets of these momenta with H_{tot}. Immediately this leads to the *secondary constraints* [20]

$$\begin{aligned} \tilde{\mathcal{H}} &= 0 && (29a) \\ \tilde{\chi}^a &= 0 && (29b) \\ \tilde{\chi}^{ab} &= 0. && (29c) \end{aligned}$$

Consequently, the Hamiltonian vanishes on the constraint surface.

The new constraints (29a)-(29c) also have to consistent with the equations of motion. So their Poisson brackets with H_{tot} must vanish on the constraint surface, or else there will be further constraints. Calculating Poisson brackets with H_{tot} reduces to working out the pair-wise brackets between the quantities $\tilde{\mathcal{H}}$, $\tilde{\chi}^a$ and $\tilde{\chi}^{ab}$. The non-vanishing brackets between the canonical variables are:

$$\begin{aligned} \left[\pi, \alpha'\right] &= \delta\left(x, x'\right) \qquad \left[\pi^a, \beta'_b\right] = \delta^a{}_b \delta\left(x, x'\right) \qquad \left[B_{ab}, p'^{cd}\right] = \delta^{[cd]}{}_{ab}\delta\left(x, x'\right) \\ \left[\gamma_{ab}, \pi'^{cd}\right] &= \delta^{(cd)}{}_{ab}\delta\left(x, x'\right) \qquad \left[A_{abc}, \pi'^{def}\right] = \delta^{[def]}{}_{abc}\delta\left(x, x'\right) \end{aligned}$$

Here $'$ means that a quantity is evaluated at x', $\delta\left(x, x'\right)$ is the 10-dimensional delta function and $\delta^{a_1 \ldots a_k}_{b_1 \ldots b_k} = \delta^{a_1}_{b_1} \ldots \delta^{a_k}_{b_k}$.

Before proceeding to the derivation of the Poisson brackets, we note that in general, the brackets are expressed in terms of generalized functions - δ-functions and their derivatives. So the technically correct way to handle them is to introduce arbitrary test functions and consider the action of the generalized function on them.

The calculation can be simplified if we notice the following. For an arbitrary Λ_{ab}, we have

$$\begin{aligned} \delta_\Lambda A &= \left[A_{def}, \int \tilde{\chi}'^{ab}\Lambda'_{ab} d^{10}x'\right] = -3\int \left[A_{def}, \pi'^{abc}{}_{,c'}\right]\Lambda'_{ab} d^{10}x' \\ &= 3\Lambda_{[de,f]} && (30) \end{aligned}$$

So this implies that $\tilde{\chi}^{ab}$ is the generator of the gauge transformation

$$\delta_\Lambda A = d\Lambda, \tag{31}$$

and hence

$$\delta_\Lambda F = 0. \tag{32}$$

Under this transformation, we have

$$\begin{aligned} \delta_\Lambda \pi^{def} &= \left[\pi^{def}, \int \tilde{\chi}'^{ab}\Lambda'_{ab} d^{10}x'\right] \\ &= \left[\pi^{def}, \int \frac{4}{12^3}\eta^{abc_3 \ldots c_{10}} \partial'_{[c_3} A'_{c_4 c_5 c_6]} \partial'_{[c_7} A'_{c_8 c_9 c_{10}]} \Lambda'_{ab} d^{10}x'\right] \\ &= -\frac{2}{12^3}\eta^{defabcg_3 \ldots g_6} \partial_a \Lambda_{bc} F_{g_3 \ldots g_6} && (33) \end{aligned}$$

and hence

$$\delta_\Lambda \tilde{\pi}^{abc} = 0. \tag{34}$$

Using (32) and (34), it immediately follows that all brackets involving $\tilde{\chi}^{ab}$ vanish identically, since relevant terms in each constraint involve only F and $\tilde{\pi}^{abc}$.

Consider the brackets with $\tilde{\chi}_a$ now. After some index manipulation it is possible to rewrite $\tilde{\chi}_a$ as

$$\tilde{\chi}_a = \chi_a + F_{abcd}\pi^{bcd} - A_{abc}\tilde{\chi}^{bc} - 3A_{abc}\pi^{bcd}{}_{,d}. \tag{35}$$

However, $\tilde{\chi}^{bc}$ is also a constraint and moreover all its brackets with other constraints vanish, so we can replace $\tilde{\chi}_a$ by an irreducible constraint $\hat{\chi}_a$ given by

$$\hat{\chi}_a = \chi_a + F_{abcd}\pi^{bcd} - 3A_{abc}\pi^{bcd}{}_{,d} \tag{36}$$

It is hence enough to work out the brackets with $\hat{\chi}_a$.

In pure gravity, we know from [3] that χ_a generates spatial translations. Hence $\hat{\chi}_a$ acts on γ_{ab} and π^{ab} as a Lie derivative, that is, for arbitrary ξ^a

$$\begin{aligned}\left[\gamma_{mn}, \int \hat{\chi}'_a \xi'^a d^{10}x'\right] &= \mathcal{L}_\xi \gamma_{mn} \\ \left[\pi^{mn}, \int \hat{\chi}'_a \xi'^a d^{10}x'\right] &= \mathcal{L}_\xi \pi^{mn}\end{aligned}$$

We can now work out the action of $\hat{\chi}_a$ on A_{mnp} and π^{mnp}.

$$\left[A_{bcd}, \int \hat{\chi}'_a \xi'^a d^{10}x'\right] = \xi^a F_{abcd} + 3\partial_{[b}\left(\xi^a A_{|a|cd]}\right) = \mathcal{L}_\xi A_{bcd} \tag{37}$$

since A_{bcd} is a 3-form. For π^{bcd} we have

$$\begin{aligned}\left[\pi^{bcd}, \int \hat{\chi}'_a \xi'^a d^{10}x'\right] &= \partial_a\left(\pi^{bcd}\xi^a\right) - 3\partial_a\left(\xi^{[b}\pi^{|a|cd]}\right) + 3\xi^{[b}\pi^{cd]a}{}_{,a} \\ &= \mathcal{L}_\xi \pi^{bcd}\end{aligned} \tag{38}$$

since π^{bcd} is a tensor density of weight 1. Therefore $-\hat{\chi}_a$ generates spatial translations, and hence $\hat{\chi}_a$ acts as a Lie derivative. Noting that $\hat{\chi}_b$ is a covector and $\tilde{\mathcal{H}}$ is a scalar density of weight 1, we immediately see that

$$\left[\hat{\chi}_b, \int \hat{\chi}'_a \xi'^a d^{10}x'\right] = \mathcal{L}_\xi \hat{\chi}_b = \left(\xi^c \hat{\chi}_b\right)_{,c} + \hat{\chi}_c \xi^c_{,b} \tag{39}$$

$$\left[\tilde{\mathcal{H}}, \int \hat{\chi}'_a \xi'^a d^{10}x'\right] = \mathcal{L}_\xi \tilde{\mathcal{H}} = \left(\tilde{\mathcal{H}}\xi^c\right)_{,c}. \tag{40}$$

Introducing new test functions σ^a and σ, respectively, we have

$$\begin{aligned}\int\int \left[\hat{\chi}_b, \hat{\chi}'_a\right] \sigma^b \xi'^a d^{10}x d^{10}x' &= \int \hat{\chi}_c\left(\xi^c_{,b}\sigma^b - \xi^b \sigma^c_{,b}\right) d^{10}x \\ \int\int \left[\tilde{\mathcal{H}}, \hat{\chi}'_a\right] \sigma \xi'^a d^{10}x d^{10}x' &= -\int \tilde{\mathcal{H}} \xi^c \sigma_{,c} d^{10}x\end{aligned}$$

after integration by parts. This gives that these brackets vanish on the constraint surface.

We are now only left with the bracket $\left[\tilde{\mathcal{H}}, \tilde{\mathcal{H}}'\right]$. From [3], we already know $[\mathcal{H}, \mathcal{H}']$, so only need to work out the brackets $\left[F^2, \tilde{\pi}^2\right]$ and $\left[\tilde{\pi}^2, \tilde{\pi}^2\right]$, since the other cross-terms vanish. After some lengthy calculations, which are given in the Appendix, we find that

$$\left[\tilde{\mathcal{H}}, \tilde{\mathcal{H}}'\right] = 2\tilde{\chi}^a \delta_{,a}\left(x, x'\right) + \tilde{\chi}^a{}_{,a}\delta\left(x, x'\right) \tag{41}$$

which is analogous to the untilded expression for $[\mathcal{H}, \mathcal{H}']$ in [3]. In particular, $\left[\tilde{\mathcal{H}}, \tilde{\mathcal{H}}'\right]$ vanishes on the constraint surface.

Hence the full canonical description of the bosonic sector of 11-dimensional supergravity involves only three primary constraints (19a)-(19c) and the three corresponding secondary constraints (29a)-(29c). These constraints are *first-class constraints* - that is, their pairwise brackets vanish on the constraint surface, and they generate gauge transformations [20].

Consider now the quantization of this system. Adopting the same view as in [3], we will take it that any two field operators taken at the same space-time point commute. This way, the classical consistency conditions carry over to the quantum case without anomalies. So we can perform Dirac quantization [20] of the system. The constraints then become conditions on the wavefunction Ψ:

$$\begin{aligned} \pi\Psi &= 0 \quad \pi^a\Psi = 0 \quad p^{ab}\Psi = 0 \\ \tilde{\mathcal{H}}\Psi &= 0 \quad \hat{\chi}^a\Psi = 0 \quad \tilde{\chi}^{ab}\Psi = 0 \end{aligned} \tag{42a}$$

This implies that $H_{tot}\Psi = 0$, and hence from the Schrödinger equation, $\partial\Psi/\partial t = 0$.

Using the representation

$$\pi = -i\frac{\delta}{\delta\alpha} \quad \pi^a = -i\frac{\delta}{\delta\beta_a} \quad p^{ab} = -i\frac{\delta}{\delta B_{ab}} \tag{43a}$$

$$\pi^{ab} = -i\frac{\delta}{\delta\gamma_{ab}} \quad \pi^{abc} = -i\frac{\delta}{\delta A_{abc}}, \tag{43b}$$

the constraints (42a) imply that Ψ is independent of α, β_a and B_{ab}, while the constraints (42a) completely describe the dynamics of Ψ.

3 Minisuperspace

In general the wavefunction Ψ is a function on the infinite-dimensional superspace which consists of $\gamma_{ab}(x)$ and $A_{abc}(x)$ modulo diffeomorphisms and form gauge transformations. Behaviour in this infinite-dimensional space is difficult to describe, so it is useful to reduce the number of variables, by fixing some degrees of freedom. This way the infinite-dimensional superspace is reduced to a finite-dimensional minisuperspace.

To reduce the number of degrees of freedom in the metric, we consider the following ansatz for the 11-dimensional spacetime metric:

$$ds_{11}^2 = -\alpha(t)^2 dt^2 + e^{2X_1(t)}d\Omega_1^2 + ... + e^{2X_n(t)}d\Omega_n^2. \tag{44}$$

Here each $d\Omega_i^2$ is the metric of a maximally symmetric a_i-dimensional space with radius of curvature ±1 or 0. Since the space-time is 11-dimensional, we also have a condition $a_1+...+a_n = 10$. For each i, e^{X_i} is the scale factor of each spatial component. Thus the only remaining degrees of freedom which remain from γ_{ab} are the X_i.

For definiteness, suppose $a_1 = 3$ and consider the following ansatz for the 4-form:

$$F_{\alpha\beta\gamma\delta} = \dot{f}(t)\,\hat{\varepsilon}_{\alpha\beta\gamma\delta}. \tag{45a}$$

$$F_{\mu\nu\rho\sigma} = 0 \text{ otherwise} \tag{45b}$$

where $\hat{\varepsilon}_{\alpha\beta\gamma\delta}$ is the volume form on the 4-space with metric $ds^2 = -dt^2 + d\Omega_1^2$. A similar ansatz has been used in [21]. With this ansatz, the degrees of freedom A_{abc} are reduced to just $f(t)$.

The second fundamental form K_{ab} is given in this case by

$$K_{ab} = -\frac{1}{2}\alpha^{-1}\dot{\gamma}_{ab}. \tag{46}$$

From the metric ansatz, we immediately get

$$\begin{aligned} K_{ab}K^{ab} &= \alpha^{-2}\left(a_1\dot{X}_1^2 + ... + a_n\dot{X}_n^2\right) \\ K^2 &= \alpha^{-2}\dot{V}^2. \end{aligned}$$

where we have defined $V = a_1X_1 + ... + a_nX_n$. Hence we have

$$\gamma^{\frac{1}{2}} = \hat{\gamma}^{\frac{1}{2}}e^V$$

where $\hat{\gamma} = \det(\hat{\gamma}_{ab})$ is the determinant of the normalized spatial metric $\hat{\gamma}_{ab}$.

With the ansatz (45a) for the 4-form, the Chern-Simons term in the action (1) vanishes and the F^2 term becomes

$$\frac{1}{48}F^{\mu_1...\mu_4}F_{\mu_1...\mu_4} = -\frac{1}{2}\alpha^{-2}e^{-2a_1X_1}\dot{f}^2.$$

If we assume spatial sections of finite volume, for simplicity we can normalize this volume to be unity. Thus, rewriting the action in terms of the new variables f and X_i, and integrating out the spatial integral, we obtain the action for the minisuperspace model S_{mss}:

$$S_{mss} = \int dt\alpha^{-1}e^V\left[\left(a_1\dot{X}_1^2 + ... + a_n\dot{X}_n^2\right) - \dot{V}^2 + \frac{1}{2}e^{-2a_1X_1}\dot{f}^2 + \alpha^2R^{(10)}\right]. \tag{47}$$

It can be shown explicitly that the equations of motion which are obtained from this action are equivalent to the equations obtained when our ansätze for the metric and the 4-form are substituted into the full field equations for supergravity. In particular, note that the equation of motion for f is

$$\frac{d}{dt}\left(\alpha^{-1}e^{V-2a_1X_1}\dot{f}\right) = 0. \tag{48}$$

The field equation for the 4-form is

$$\begin{aligned} \nabla_\mu F^{\mu\nu\rho\sigma} &= \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}F^{\mu\nu\rho\sigma}\right) \\ &= \alpha^{-1}e^{-V}\partial_0\left(\alpha e^V F^{0\nu\rho\sigma}\right) + \frac{1}{\sqrt{g_1}}\partial_a\left(g_1^{\frac{1}{2}}F^{a\nu\rho\sigma}\right) = 0 \end{aligned}$$

where g_1 is the determinant of the metric $d\Omega_1^2$ for the 3-space. The second term in the sum vanishes due to the ansatz (45a), and the remaining equation is precisely equivalent to (48).

From our metric ansatz (44), the general form of the spatial Ricci scalar $R^{(10)}$ is

$$R^{(10)} = \sum k_i a_i (a_i - 1) e^{-2X_i}$$

where $k_i = \pm 1$ or 0. However, we will only consider the case where all but one k_i vanish. In particular, this special case encompasses the scenario where the external 4-dimensional spacetime has a Friedmann-Robertson-Walker metric with $k = -1, 0, +1$, and the 7-dimensional internal space is a Ricci-flat compact manifold. This case is of interest from a cosmological point of view and also from the point of view of M-theory special holonomy compactifications. With this, the action (47) becomes

$$S_{mss} = \int dt\alpha^{-1}e^V\left[\left(a_1\dot{X}_1^2 + ... + a_n\dot{X}_n^2\right) - \dot{V}^2 + \frac{1}{2}e^{-2a_1X_1}\dot{f}^2 + \alpha^2k_ia_i(a_i - 1)e^{-2X_i}\right]. \tag{49}$$

To get rid of the e^V factor in the integrand, it is useful to introduce a new time parameter τ which is related to t by

$$\frac{dt}{d\tau} = e^V. \tag{50}$$

Then denoting a derivative with respect to τ with a prime, for any function $G(t)$,

$$\dot{G} = e^{-V} G'$$

and so the action is now

$$S_{mss} = \int d\tau \alpha^{-1} \left[\left(a_1 X_1'^2 + ... + a_n X_n'^2 \right) - V'^2 + \frac{1}{2} e^{-2a_1 X_1} f'^2 + \alpha^2 k_i a_i (a_i - 1) e^{2(V - X_i)} \right] \quad (51)$$

In the integrand of (49) we have a quadratic form in the X_i', but as with any quadratic form it is always possible to diagonalise it. We would like to obtain a Hamiltonian which as simple as possible, so that it would be possible to solve the classical and quantum equation explicitly. For this, the exponentials in the integrand of (51) should only depend on a single variable each. This poses a problem if $i = 1$, because in general, this quadratic form will not be diagonal if we have X_1 and $V - X_1$ as independent variables. Therefore, if we want to have $i = 1$, and still be able to get explicit solutions, we have to sacrifice the f term which comes from the 4-form. So in this case one of our new variables will be $V - X_1$. Conversely, if we want to keep this 4-form term, we need $i \neq 1$, and thus treat X_1 and $V - X_i$ as independent variables.

So we will consider two cases - one where the 4-form is turned off, and i is arbitrary. In this case X_1 is no longer special, and in fact, we do not necessarily need $a_1 = 3$ - we only need $a_1 \neq 1$, so by setting $i = 1$ we do not lose any generality. The second case is where the f term is present, and $i \neq 1$. Without loss of generality we set $i = n$.

In the first case, we will make the following change of variables

$$\begin{aligned} Y_1 &= (a_1 - 1) X_1 + a_2 X_2 + ... + a_n X_n \\ Y_2 &= (a_1 + a_2 - 1) X_2 + a_3 X_3 + ... + a_n X_n \\ Y_3 &= (a_1 + a_2 + a_3 - 1) X_3 + a_4 X_4 + ... + a_n X_n \\ & ... \\ Y_n &= (a_1 + a_2 + ... + a_n - 1) X_n. \end{aligned}$$

Then if we define the coefficients b_i by

$$\begin{aligned} b_1^2 &= a_1 (a_1 - 1)^{-1} \\ b_2^2 &= a_2 (a_1 - 1)^{-1} (a_1 + a_2 - 1)^{-1} \\ & ... \\ b_n^2 &= a_n (a_1 + ... + a_{n-1} - 1)^{-1} (a_1 + ... + a_n - 1)^{-1}, \end{aligned}$$

we can write

$$\left(a_1 X_1'^2 + ... + a_n X_n'^2 \right) - V'^2 = -b_1^2 Y_1'^2 + b_2^2 Y_2'^2 + ... + b_n^2 Y_n'^2 \quad (52)$$

and moreover

$$V = b_1^2 Y_1 - b_2^2 Y_2 - ... - b_n^2 Y_n. \quad (53)$$

With these variables, and $f' = 0$, the action (51) becomes

$$S_{mss} = \int d\tau \left[\alpha^{-1} \left(-b_1^2 Y_1'^2 + b_2^2 Y_2'^2 + ... + b_n^2 Y_n'^2 \right) + \alpha \frac{1}{4} K^2 e^{2Y_1} \right] \quad (54)$$

where $K^2 = 4k_1 a_1 (a_1 - 1)$.

Now let us consider the second case, with the non-trivial 4-form. Here we need $V - X_n$ and X_1 to be independent variables. Notice that Y_n is proportional to X_n. So if in the definitions for Y_1 and Y_n we replace X_1 with X_n and vice versa, and a_1 with a_n, and vice versa, we get variables which perfectly fit our needs. The other variables can obviously remain unchanged, but we relabel them for convenience. Therefore, overall we get the following set of variables:

$$\begin{aligned}
Z_1 &= (a_1 + a_2 + ... + a_n - 1)\, X_1 \\
Z_2 &= a_1 X_1 + (a_2 + a_3 + ... + a_n - 1)\, X_2 \\
&\ ... \\
Z_{n-2} &= a_1 X_1 + a_2 X_2 + ... + a_{n-3} X_{n-3} + (a_{n-2} + a_{n-1} + a_n - 1)\, X_{n-2} \\
Z_{n-1} &= (a_n + a_{n-1} - 1)\, X_{n-1} + a_{n-2} X_{n-2} + ... + a_1 X_1 \\
Z_n &= (a_n - 1)\, X_n + a_{n-1} X_{n-1} + ... + a_1 X_1 .
\end{aligned}$$

Then if we define coefficients c_i^2 by

$$\begin{aligned}
c_1^2 &= a_1 (a_2 + a_3 + ... + a_n - 1)^{-1} (a_1 + a_2 + ... + a_n - 1)^{-1} \\
c_2^2 &= a_2 (a_3 + a_4 + ... + a_n - 1)^{-1} (a_2 + a_3 + ... + a_n - 1)^{-1} \\
&\ ... \\
c_{n-1}^2 &= a_{n-1} (a_n - 1)^{-1} (a_{n-1} + a_n - 1)^{-1} \\
c_n^2 &= a_n (a_n - 1)^{-1} ,
\end{aligned}$$

we can write

$$\left(a_1 X_1'^2 + ... + a_n X_n'^2\right) - V'^2 = c_1^2 Z_1'^2 + ... + c_{n-1}^2 Z_{n-1}'^2 - c_n^2 Z_n'^2 \tag{55}$$

and moreover

$$V = -c_1^2 Z_1 - ... - c_{n-1}^2 Z_{n-1} + c_n^2 Z_n \tag{56}$$

We have assumed that $a_n \neq 1$. Also noting that $a_1 = 3$, and $a_n + a_{n-1} + ... + a_1 = 10$, the action is now written as

$$S_{mss} = \int d\tau \left[\alpha^{-1} \left(c_1^2 Z_1'^2 + ... + c_{n-1}^2 Z_{n-1}'^2 - c_n^2 Z_n'^2 + \frac{1}{2} e^{-\frac{2}{3} Z_1} f'^2 \right) + \alpha \frac{1}{4} K^2 e^{2 Z_n} \right] . \tag{57}$$

Here $K^2 = 4 k_n a_n (a_n - 1)$.

4 Minisuperspace solutions with trivial 4-form

Here we consider the solutions with a trivial 4-form. Using the action (54), we shall construct the canonical formulation of this model. The Lagrangian L_{mss} is given by

$$L_{mss} = \alpha^{-1} \left(-b_1^2 Y_1'^2 + b_2^2 Y_2'^2 + ... + b_n^2 Y_n'^2 \right) + \alpha \frac{1}{4} K^2 e^{2 Y_1} . \tag{58}$$

The canonical variables here are $Y_1, Y_2, ..., Y_n$ and α, so the conjugate momenta are

$$\begin{aligned}
p_1 &= \frac{\partial L_{mss}}{\partial Y_1'} = -2\alpha^{-1} b_1^2 Y_1' \\
p_i &= \frac{\partial L_{mss}}{\partial Y_i'} = 2\alpha^{-1} b_i^2 Y_i' \\
p_\alpha &= \frac{\partial L_{mss}}{\partial \alpha'} = 0
\end{aligned}$$

We can now write down the Hamiltonian:

$$H_{mss} = \frac{1}{4} \alpha \left(-b_1^{-2} p_1^2 + b_2^{-2} p_2^2 + ... + b_n^{-2} p_n^2 \right) - \alpha \frac{1}{4} K^2 e^{2 Y_1} \tag{59}$$

From this, the canonical action becomes simply

$$S=\int\left(p_1Y_1'+...+p_nY_n'-\alpha\tilde{\mathcal{H}}\right)d\tau$$

where $\tilde{\mathcal{H}}$ is given by

$$\tilde{\mathcal{H}}=\frac{1}{4}\left(-b_1^{-2}p_1^2+b_2^{-2}p_2^2+...+b_n^{-2}p_n^2\right)-\frac{1}{4}K^2e^{2Y_1}. \tag{60}$$

By varying S with respect to α, we obtain the constraint $\tilde{\mathcal{H}}=0$. This is precisely the special case of the Hamiltonian constraint (29a). Note that with our ansätze, the other two constraints (29b) and (29c) are trivial. As we know from the general case, α is arbitrary, so for convenience we will set the gauge $\alpha=1$.

In the gauge $\alpha=1$, the classical equations are thus

$$\begin{array}{cc} p_1'=\frac{1}{2}K^2e^{2Y_1} & Y_1'=-\frac{1}{2}b_1^{-2}p_1 \\ p_i'=0 & Y_i'=\frac{1}{2}b_i^{-2}p_i \end{array} \tag{61}$$

where $i=2,...,n$. Since the "potential" does not depend on Y_i, we get that the p_i are constant in τ, and thus we have

$$Y_i=\frac{1}{2}b_i^{-2}p_i\tau+Y_i(0)$$

From (53), the volume factor e^V is given by

$$\begin{aligned} e^V &= \exp\left(b_1^2Y_1-b_2^2Y_2-...-b_n^2Y_n\right) \\ &= A\exp\left(b_1^2Y_1\right)\exp\left[-\frac{1}{2}\left(p_2+...+p_n\right)\tau\right] \end{aligned} \tag{62}$$

where $A=\exp\left[b_2^2Y_2(0)+...+b_n^2Y_n(0)\right]$, so that from (50),

$$\frac{dt}{d\tau}=Ae^{b_1^2Y_1}e^{-\frac{1}{2}p_s\tau} \tag{63}$$

where $p_s=p_2+...+p_n$. Thus the original time variable t can be restored in terms of τ once $Y_1(\tau)$ is known. Since all momenta except p_1 are constant in τ, we can rewrite the Hamiltonian constraint (60) as

$$b_1^2K^2e^{2Y_1}=\xi^2-p_1^2 \tag{64}$$

where ξ is a constant given by

$$\xi^2=b_1^2\left(b_2^{-2}p_2^2+...+b_n^{-2}p_n^2\right). \tag{65}$$

Using (64), the equation of motion for p_1 becomes

$$p_1'=\frac{1}{2}b_1^{-2}\left(\xi^2-p_1^2\right). \tag{66}$$

We have thus seen that after a reparametrization of the time coordinates, and a change of variables on the minisuperspace, the classical minisuperspace system is described by equations (64) and (66). Essentially these are equations of motion of a particle moving in the potential $-\frac{1}{4}K^2e^{2Y_1}$ constrained so that the total energy vanishes. Apart from the initial conditions, the solutions depend on the parameters b_i^2 (which are determined by the dimensionalities of the spatial components of space-time) and the curvature parameter K^2. In fact, from (64) we see that the sign of K^2 affects the nature of equation (66) and hence the qualitative behaviour of the solution.

A similar system is considered in [22]-[24], where the dynamics of scale factors is studied in the presence of wall potentials near a cosmological singularity, giving rise to "cosmological billiards".

We now proceed to the quantization of the minisuperspace model. The canonical variables in the minisuperspace are now Y_i for $i = 1, ..., n$, and the corresponding momenta p_i for $i = 1, ..., n$. For the momentum operators, we use the following representation

$$p_i = -i\frac{\partial}{\partial Y_i}$$

The dynamics of the wavefunction are governed only by the constraint $\tilde{\mathcal{H}}\Psi = 0$. In our representation this gives the equation

$$\left(-b_1^{-2}\frac{\partial^2}{\partial Y_1^2} + b_2^{-2}\frac{\partial^2}{\partial Y_2^2} + ... + b_n^{-2}\frac{\partial^2}{\partial Y_n^2}\right)\Psi + K^2 e^{2Y_1}\Psi = 0. \tag{67}$$

This is the Wheeler-DeWitt equation [3] for our minisuperspace model. As always when passing from a classical expression to a quantum one there are issues of factor ordering. However the qualitative behaviour of a differential equation is mainly determined by the principal symbol, and since in our parametrization the metric on the minisuperspace is flat, we choose the ordering which allows to write down an exact solution.

This equation separates, and we get

$$\frac{\partial^2 G}{\partial Y_1^2} - \left(b_1^2 K^2 e^{2Y_1} - k_1^2\right) G = 0 \tag{68}$$

where

$$\Psi = e^{ik_2Y_2}...e^{ik_nY_n}G\left(Y_1\right).$$

Here k_1 is given by

$$k_1^2 = b_1^2\left(b_2^{-2}k_2^2 + ... + b_n^{-2}k_n^2\right) \tag{69}$$

and the k_i for $i \geq 2$ are eigenvalues of the momenta p_i. Note that (68) is the precise quantum analogue of the classical constraint (64). Moreover, it can be viewed as a one-dimensional Schrödinger equation with an exponential potential $b_1^2K^2e^{2Y_1}$. Setting $z = e^{Y_1}$, we get

$$z^2\frac{\partial^2 G}{\partial z^2} + z\frac{\partial G}{\partial z} - \left(K^2 b_1^2 z^2 - k_1^2\right) G = 0 \tag{70}$$

therefore up to rescaling of variables, this is a Bessel equation. Depending on the sign of K^2, solutions can be expressed in terms different types of Bessel functions.

4.1 Case 1: $K^2 = 0$

Suppose the spatial curvature fully vanishes, so that $K^2 = 0$. Classically this gives that p_1 is also constant, and Y_1 is given by

$$Y_1 = -\frac{1}{2}b_1^{-2}p_1\tau + Y_1\left(0\right)$$

From (64) we see that $p_1^2 = \xi^2$.

Note that from (63) that

$$t = -\frac{2A}{p_1 + p_s}e^{-\frac{1}{2}(p_1+p_s)\tau}$$

So depending on the sign of $p_1 + p_s$, t is either always positive for all values of τ or always negative for all values of τ. Overall, this can be regarded as a generalization of the Kasner metric.

In the quantum case, variables separate trivially, and we get

$$\Psi = N e^{ik_1 Y_1} ... e^{ik_n Y_n} e^{ik_f f}$$

where

$$-b_1^{-2} k_1^2 + b_2^{-2} k_2^2 + ... + b_n^{-2} k_n^2 = 0.$$

4.2 Case 2: $K^2 > 0$

Suppose $K^2 > 0$. From the constraint (64) we see that we must have $|p_1| < \xi$. From (66), and using the condition on p_1, we get

$$\frac{1}{2} b_1^{-2} \int d\tau = \int \frac{dp_1}{\xi^2 - p_1^2} = \xi^{-1} \operatorname{arctanh}\left(\xi^{-1} p_1\right)$$

Hence

$$p_1 = \xi \tanh\left(\frac{1}{2} b_1^{-2} \xi \tau + \tau_0\right) \tag{71}$$

Now Y_1 is determined by

$$Y_1' = -\frac{1}{2} b_1^{-2} \xi \tanh\left(\frac{1}{2} b_1^{-2} \xi \tau + \tau_0\right)$$

So

$$Y_1 = c_1 - \log\left(\cosh\left(\frac{1}{2} b_1^{-2} \xi \tau + \tau_0\right)\right)$$

The relation (64) fixes the constant c_1, hence Y_1 is given by

$$Y_1 = -\log\left(K b_1 \xi^{-1} \cosh\left(\frac{1}{2} b_1^{-2} \xi \tau + \tau_0\right)\right) \tag{72}$$

Figure 1 shows the behavior in phase space (with $\tau_0 = 0$). We can see that this solution has only one branch - the negative and positive momentum sectors are smoothly connected.

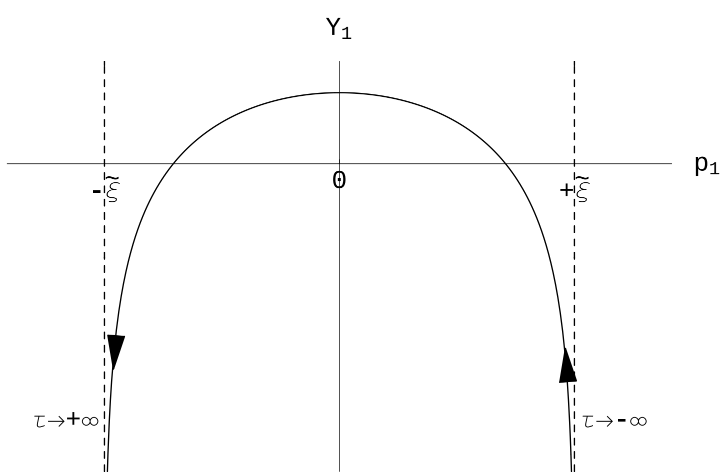

Figure 1: Phase space behaviour for $K^2 > 0$

We have the following asymptotic behaviour for Y_1 and p_1:

$$\begin{array}{lll} \tau \longrightarrow +\infty & p_1 \longrightarrow \xi & Y_1 \sim -\frac{1}{2} b_1^{-2} \xi \tau = -\frac{1}{2} b_1^{-2} p_1 \tau \\ \tau \longrightarrow -\infty & p_1 \longrightarrow -\xi & Y_1 \sim +\frac{1}{2} b_1^{-2} \xi \tau = -\frac{1}{2} b_1^{-2} p_1 \tau \end{array} \tag{73}$$

Let us now investigate the behaviour of the original scale factors X_i. From definition of Y_1, $X_1 = V - Y_1$. So let us first look at the asymptotic behaviour of V. From (53), up to a constant we have

$$V = b_1^2 Y_1 - \frac{1}{2} p_s \tau.$$

Thus from the asymptotic behaviour of Y_1 (73), as $\tau \longrightarrow \pm\infty$ we get

$$V \sim -\frac{1}{2} \tau \left(p_s \pm \xi\right). \tag{74}$$

Note that $p_s^2 - \xi^2 \leq 0$, but since $\xi > 0$, we have $p_s - \xi < 0$ and $p_s + \xi > 0$. Therefore, $V \longrightarrow -\infty$ as $\tau \longrightarrow \pm\infty$, so in fact V has very similar asymptotic behaviour to Y_1. From (74), the behaviour of X_1 is easily obtained:

$$\begin{aligned} X_1 &\sim -\frac{1}{2} \tau \left(p_s \pm \left(1 - b_1^{-2}\right) \xi\right) \\ &= -\frac{1}{2} \tau \left(p_s \pm \frac{1}{a_1} \xi\right) \end{aligned}$$

It follows that the qualitative behaviour of X_1 does actually depend on the numerical values of the constant momenta p_i. It can easily be seen now, that all other X_i will also be asymptotically proportional to τ, but with different constants of proportionality which also depend on the initial conditions.

By construction, the overall 11-dimensional space is Ricci-flat. However let us look at what happens to the intrinsic curvature from 4-dimensional point of view. The expression for the 4-dimensional Ricci scalar is given by

$$R^{(4)} = \frac{1}{4} K^2 e^{-2X_1} + \left(a_1 - 1\right) \ddot{X}_1 + a_1 \left(a_1 - 1\right) \dot{X}_1^2.$$

After changing variables and the time parameter, and applying the constraint and equations of motion, we get

$$R^{(4)} = -\frac{1}{4} e^{-2V} \left(a_1 - 1\right) \left(a_1^{-2} p_1^2 - a_1^{-2} \xi^2 - p_s b_1^{-2} p_1 - p_s^2 \left(a_1 - 1\right)\right). \tag{75}$$

Since p_1 is always bounded (71), the curvature blows up when $V \longrightarrow -\infty$, and as we know this does happen when $\tau \longrightarrow \pm\infty$. So although the 11-dimensional space is flat, from the 4-dimensional point of view there is a curvature singularity.

The solutions we had so far were in terms of the time parameter τ. To relate it to the original time parameter t, we need to integrate e^V. In this case

$$e^V = A \left[K^{-1} b_1^{-1} \xi \operatorname{sech}\left(\frac{1}{2} b_1^{-2} \xi \tau + \tau_0\right)\right]^{b_1^2} e^{-\frac{1}{2} p_s \tau}$$

where $p_s = p_2 + ... + p_n$. For $c_0 = 0$, the integral of this expression can be evaluated explicitly in terms of the hypergeometric function ${}_2F_1\left(a, b; c; z\right)$ [25]:

$$t\left(\tau\right) = A \left(2K^{-1} b_1^{-1} \xi\right)^{b_1^2} \frac{2}{\xi - p_s} e^{\frac{1}{2}\tau\left(\xi - p_s\right)} {}_2F_1 \left(b_1^2, \frac{b_1^2}{2\xi}\left(\xi - p_s\right); 1 + \frac{b_1^2}{2\xi}\left(\xi - p_s\right); -e^{b_1^{-2}\xi\tau}\right)$$

From this we can at least extract asymptotic behaviour of t as $\tau \longrightarrow \pm\infty$ [25],

$$t \sim t_0^{\pm} - \frac{2A\left(2K^{-1}b_1^{-1}\xi\right)^{b_1^2}}{p_s \pm \xi} e^{-\frac{1}{2}\tau(p_s \pm \xi)} \tag{76}$$

where $t_0^{\pm}$ are constants, which we can choose such that $t_0^- = 0$. This behaviour is hence similar to the $K^2 = 0$ case for $p_1 = \pm\xi$. We know that $p_s - \xi < 0$ and $p_s + \xi > 0$. Therefore as $\tau \longrightarrow -\infty$, the time parameter t approaches 0 from above and as $\tau \longrightarrow +\infty$, t approaches t_0^+ from below.

Hence overall, at small t, the overall size of the universe is very small, and the 4-dimensional curvature is very high, then as t increases, the size of the universe increases and hence the curvature decreases up to a point, after which the universe collapses again and the curvature blows up within a finite time t_0^+.

Now consider the quantized system. For positive K^2, the Wheeler-DeWitt equation (70) becomes a modified Bessel's equation with an imaginary parameter ik_1. So the solutions are linear combinations of modified Bessel functions of first and second kind - $I_{ik_1}(z)$ and $K_{ik_1}(z)$ respectively. If we impose the condition $|\Psi| < \infty$ on the wavefunction, this uniquely selects $K_{ik}(z)$ [25]. This choice selects the the wavefunction which decays as $Y_1 \longrightarrow +\infty$, which is consistent with the exponential wall potential in (68).

Thus up to a normalization factor, the wavefunction is

$$\Psi = e^{ik_2Y_2}...e^{ik_nY_n} K_{ik_1}\left(Kb_1e^{Y_1}\right). \tag{77}$$

A general property of $K_{ik_1}(z)$ is that for $0 < z < |k_1|$, the is oscillatory, and for $z > |k_1|$, the function decays with asymptotic behaviour as $z \longrightarrow \infty$ given by

$$K_{ik_1}(z) \sim e^{(k-1-i)\frac{\pi}{2}} \sqrt{\frac{\pi}{2}} z^{-\frac{1}{2}} e^{-z}. \tag{78}$$

In our case however, $z = Kb_1e^{Y_1}$, so the wavefunction decays extremely fast for $z > |k_1|$. From the classical solution (72),

$$z = Kb_1e^{Y_1} \leq \xi,$$

so the region of the minisuperspace where the wavefunction is oscillatory corresponds to the classically allowed region, and outside it, the wavefunction amplitude is negligibly small.

For $z \longrightarrow 0$, the asymptotic behaviour of $K_{ik_1}(z)$ is [25]:

$$K_{ik_1}(z) \sim \frac{\pi}{2k_1 \sinh(k_1\pi)} \left(\frac{\left(\frac{1}{2}z\right)^{ik_1}}{\Gamma(ik_1)} + \frac{\left(\frac{1}{2}z\right)^{-ik_1}}{\Gamma(-ik_1)} \right) \tag{79}$$

Using this, for $Y_1 \longrightarrow -\infty$, we have the following asymptotic behaviour for Ψ:

$$\begin{aligned} \Psi &\sim Ne^{ik_2Y_2}...e^{ik_nY_n} \left(\frac{\left(\frac{1}{2}Kb_1\right)^{ik_1}}{\Gamma(ik_1)} e^{ik_1Y_1} + \frac{\left(\frac{1}{2}Kb_1\right)^{-ik_1}}{\Gamma(-ik_1)} e^{-ik_1Y_1} \right) \\ &= \Psi^{(-)} + \Psi^{(+)} \end{aligned} \tag{80}$$

where N is a constant. Thus asymptotically, Ψ splits into left-moving and right-moving parts, $\Psi^{(-)}$ and $\Psi^{(+)}$ respectively, with the role of the time-like coordinate being assigned to Y_1. These plane waves move along the vector $(k_2, ..., k_n)$ in the "space-like" part of the minisuperspace. By applying the Y_1-momentum operator $p_1 = -i\frac{\partial}{\partial Y_1}$, we find that the p_1 eigenvalue for $\Psi^{(-)}$ is k_1 and the eigenvalue for $\Psi^{(+)}$ is $-k_1$. The constant k_1 corresponds to the classical quantity ξ and therefore left-movers correspond to the sector of the classical solution where $p_1 > 0$ and the right movers correspond to the sector where $p_1 < 0$. Also, from (80), we can infer that $|\Psi|$ fluctuates with amplitude $\left|\frac{N}{\Gamma(ik_1)}\right|$.

The left-moving waves can be interpreted as reflections of the right-moving waves. Effectively, such a reflection is a transition from the negative momentum sector to the positive momentum sector. In the classical system, these sectors are smoothly connected, and in the quantum system, this is manifested by the fact that the reflection coefficient between the two plane waves is $R = \left|\Psi^{(+)}\right|^2 \Big/ \left|\Psi^{(-)}\right|^2 = 1$. So in fact the two sectors are reflections of each other.

A similar behaviour was discussed in [11], in the context of a four-dimensional gravi-dilaton system with a negative, specially chosen dilaton potential. Here the smooth branch transition arises naturally from a positive curvature background since the positive curvature term in our action (51) gives rise to a negative potential in the Hamiltonian (59).

4.3 Case 3: $K^2 < 0$

Now suppose $K^2 < 0$. Letting $\tilde{K}^2 = -K^2$ we thus have from (64)

$$b_1^2 \tilde{K}^2 e^{2Y_1} = p_1^2 - \xi^2 \tag{81}$$

Therefore in this case we have $|p_1| > \xi$, so from (66), and using the condition on p_1, we get

$$\frac{1}{2} b_1^{-2} \int d\tau \int \frac{dp_1}{\xi^2 - p_1^2} = \xi^{-1} \operatorname{arccoth}\left(\xi^{-1} p_1\right)$$

Hence

$$p_1 = \xi \coth\left(\frac{1}{2} b_1^{-2} \xi \tau + \tau_0\right) \tag{82}$$

and Y_1 is given by

$$Y_1 = -\log\left(\left|\tilde{K} b_1 \xi^{-1} \sinh\left(\frac{1}{2} b_1^{-2} \xi \tau + \tau_0\right)\right|\right) \tag{83}$$

Let $\tau_0 = 0$. Then the phase space behaviour is shown in Figure 2. Now we see that there are two branches - one for which $p_1 > \xi$ and τ is positive, and one for which $p_1 < -\xi$ and τ is negative.

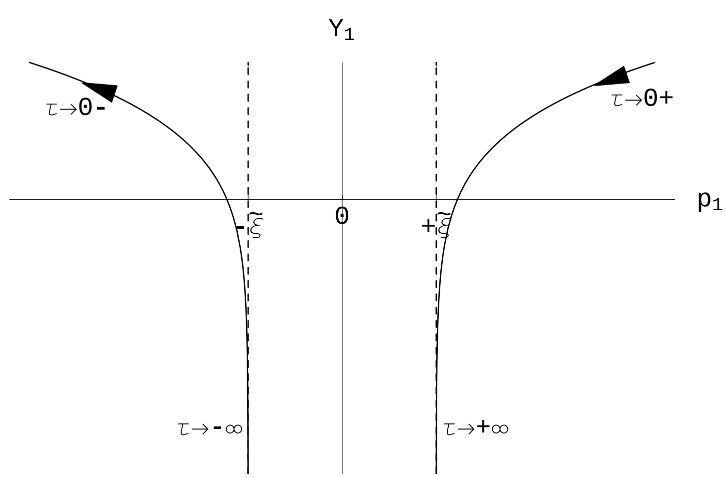

Figure 2: Phase space behaviour for $K^2 < 0$

asymptotic behaviour of Y_1, V and X_1 as $\tau \longrightarrow \pm\infty$ is the same as in the $K^2 > 0$ case, and similarly, $R^{(4)}$ blows up when $\tau \longrightarrow \pm\infty$. To give an explicit relation between time parameters t and τ, we first need

$$e^V = A\left|\tilde{K}^{-1}b_1^{-1}\xi\,\mathrm{csch}\left(\frac{1}{2}b_1^{-2}\xi\tau+\tau_0\right)\right|^{b_1^2} e^{-\frac{1}{2}p_s\tau}$$

In this case, at least for $\tau_0 = 0$, the integral can be explicitly evaluated in terms of the hypergeometric function ${}_2F_1(a,b;c,z)$. Hence have [25]

$$t(\tau) = \begin{cases} \mathrm{Re}\left((-1)^{b_1^2}2A\left(2\tilde{K}^{-1}b_1^{-1}\xi\right)^{b_1^2}\frac{e^{\frac{1}{2}\tau(\xi-p_s)}}{\xi-p_s}\,{}_2F_1\left(b_1^2,\frac{b_1^2}{2\xi}(\xi-p_s);1+\frac{b_1^2}{2\xi}(\xi-p_s);e^{b_1^{-2}\xi\tau}\right)\right) & \text{for } \tau>0\\ 2A\left(2\tilde{K}^{-1}b_1^{-1}\xi\right)^{b_1^2}\frac{e^{\frac{1}{2}\tau(\xi-p_s)}}{\xi-p_s}\,{}_2F_1\left(b_1^2,\frac{b_1^2}{2\xi}(\xi-p_s);1+\frac{b_1^2}{2\xi}(\xi-p_s);e^{b_1^{-2}\xi\tau}\right) & \text{for } \tau<0\end{cases}$$

Using asymptotic behavior of ${}_2F_1(a,b;b+1,z)$ for $z \longrightarrow \infty$, as $\tau \longrightarrow \pm\infty$ we get same behavior as in the positive curvature case [25]:

$$t \sim -\frac{2A\left(2\tilde{K}^{-1}b_1^{-1}\xi\right)^{b_1^2}}{p_s \pm \xi}e^{-\frac{1}{2}\tau(p_s\pm\xi)} + t_0^{\pm}.$$

where $t_0^{\pm}$ are constants, and again we can choose $t_0^- = 0$. As in the positive curvature case, $p_s-\xi < 0$ and $p_s+\xi > 0$. Therefore as $\tau \longrightarrow -\infty$, t approaches 0 from above and as $\tau \longrightarrow +\infty$, t approaches t_0^+ from below. However we also have this time that as $\tau \longrightarrow 0$, $|t| \longrightarrow \infty$. This implies that t_0^+ must actually be negative, and the overall behaviour is that as τ goes from $-\infty$ to 0 and t goes from 0 to $+\infty$, and as τ goes from 0 to $+\infty$ and t goes from $-\infty$ to t_0^+. Thus, unlike the $K^2 > 0$ case, t is unbounded.

Hence in this scenario, for negative t the universe collapses as $t \longrightarrow t_0^+$ and for positive t it expands. Moreover at $t = 0$ and $t = t_0^+$, the 4-dimensional curvature blows up. This is thus very similar to the "pre-Big Bang" scenario [14]. Note that the scale factor X_n is proportional to Y_n and is hence proportional to τ. But as $t \longrightarrow \pm\infty$, $\tau \longrightarrow 0$, thus the volume of this component of the internal space is stabilised as $t \longrightarrow \pm\infty$.

Consider the quantum system now. For $K^2 < 0$, the solutions of equation (70) are linear combinations of Bessel functions of the first and second kind - $J_{ik_1}(z)$ and $Y_{ik_1}(z)$ respectively, with an imaginary parameter ik_1. Alternatively, we can write the solution in terms Hankel functions $H^1_{ik_1}(z)$ and $H^2_{ik_1}(z)$. Hankel functions are defined as following:

$$\begin{aligned} H^1_\nu(z) &= J_\nu(z) + iY_\nu(z)\\ H^2_\nu(z) &= J_\nu(z) - iY_\nu(z). \end{aligned}$$

Consider the limit as $z \longrightarrow \infty$ (so that $Y_1 \longrightarrow \infty$). Then

$$H^1_{ik}(z) \sim \sqrt{\frac{2}{\pi z}}e^{\frac{k\pi}{2}}e^{-i\frac{\pi}{4}}e^{iz} \tag{84}$$

$$H^2_{ik}(z) \sim \sqrt{\frac{2}{\pi z}}e^{\frac{-k\pi}{2}}e^{i\frac{\pi}{4}}e^{-iz} \tag{85}$$

Depending on the boundary conditions, any linear combinations of these can be chosen. Boundary conditions for this type of wavefunctions have been well studied [12]. We will impose the the so-called tunneling boundary condition, to select only left-moving waves at large z so that the wavefunction up to a normalization factor is given by

$$\Psi = e^{ik_2Y_2}...e^{ik_nY_n}e^{ik_{1f}f}H^1_{ik_1}\left(\tilde{K}b_1e^{Y_1}\right).$$

Here the behaviour of the wavefunction is such that for negative Y_1, $|\Psi|$ is mostly oscillatory, while for positive Y_1, the wavefunction decays as e^{-Y_1}, which is much slower than the decay as $Y_1 \longrightarrow +\infty$ in the $K^2 > 0$ case. In the current case, all values of Y_1 are allowed classically, whereas in the $K^2 > 0$ case, Y_1 is bounded. This explains the different decay rates.

For $z \longrightarrow 0$, the asymptotic behaviour of H^1_ν is [25]:

$$H^1_\nu(z) \sim i\csc(\nu\pi)\left(e^{-\nu\pi i}\frac{\left(\frac{1}{2}z\right)^\nu}{\Gamma(1+\nu)} - \frac{\left(\frac{1}{2}z\right)^{-\nu}}{\Gamma(1-\nu)}\right) \tag{86}$$

and therefore, for $v = ik_1$, we get

$$H^1_{ik_1}(z) \sim \frac{ik_1}{\sinh(k_1\pi)}\left(e^{k_1\pi}\frac{\left(\frac{1}{2}z\right)^{ik_1}}{\Gamma(ik_1)} + \frac{\left(\frac{1}{2}z\right)^{-ik_1}}{\Gamma(-ik_1)}\right)$$

The asymptotic behaviour as $Y_1 \longrightarrow -\infty$ is given by

$$\begin{aligned}\Psi &= Ne^{ik_2Y_2}...e^{ik_nY_n}e^{ik_ff}\left(e^{k_1\pi}\frac{\left(\frac{1}{2}\tilde{K}b_1\right)^{ik_1}}{\Gamma(ik_1)}e^{ik_1Y_1} + \frac{\left(\frac{1}{2}\tilde{K}b_1\right)^{-ik_1}}{\Gamma(-ik_1)}e^{-ik_1Y_1}\right) \\ &= \Psi^{(-)} + \Psi^{(+)}\end{aligned} \tag{87}$$

where N is a constant. Interpreting Y_1 as the timelike coordinate, the wavefunction is decomposed into left and right moving waves along the vector $(k_2, ..., k_n, k_f)$ in the "space-like" part of the superspace. Note that k_1 is proportional to the magnitude of this vector.

From (87), we get that $|\Psi|$ oscillates around $\left|\frac{N}{\Gamma(ik_1)}\right| e^{k_1\pi}$ with amplitude $\left|\frac{N}{\Gamma(ik_1)}\right|$. So although the amplitude of oscillations is the same as for the case $K^2 > 0$, the fluctuation relative to the value of $|\Psi|$ is very small for large k_1. In this case the $\Psi^{(-)}$ term dominates, and $|\Psi|$ is almost constant as $Y_1 \longrightarrow -\infty$.

As in the the $K^2 > 0$ case the left moving waves correspond to the classical positive momentum, positive τ branch, and can be interpreted as being incident from the right. The right moving waves correspond to the classical negative momentum, negative τ branch, and can be interpreted as a reflection of the incident wave. The ratio of the reflected and incident amplitudes is

$$R_{k_1} = \frac{\left|\Psi^{(+)}\right|^2}{\left|\Psi^{(-)}\right|^2} = e^{-2k_1\pi}$$

and this gives the transition probability from the positive momentum branch to the negative momentum branch. But as we have seen, positive τ corresponds to negative t and vice versa. So we have a transition from classically disconnected negative time branch to the positive time branch. Thus there is finite probability of a transition from a pre-Big Bang regime to a post-Big Bang regime. This corresponds to results obtained in [11] in a string theory context with a positive dilaton potential in the Hamiltonian. Here were obtain similar behaviour, but the potential naturally comes from the spatial curvature. By choosing the boundary conditions as we did, we made sure that the transition is in the correct direction when compared with classical solutions.

Classically k_1 corresponds to ξ, and 2ξ is the distance between the two branches in the (Y_1, p_1) phase space, so can write the reflection coefficient as $R_\xi = e^{-2\xi\pi}$.

5 Minisuperspace solutions for non-trivial 4-form

Now let us go back to the full minisuperspace model with the non-trivial 4-form contribution. We basically follow same steps as in the case with the trivial 4-form. The Lagrangian in this case is

$$L_{mss} = \alpha^{-1}\left(c_1^2 Z_1'^2 + ... + c_{n-1}^2 Z_{n-1}'^2 - c_n^2 Z_n'^2 + \frac{1}{2}e^{-\frac{2}{3}Z_1} f'^2\right) + \alpha\frac{1}{4}K^2 e^{2Z_n}. \tag{88}$$

The canonical variables are $Z_1, ..., Z_n$ together with f and α. Define π_α, π_i and π_f to be momenta conjugate to α, Z_i and f respectively, so that

$$\begin{aligned}
\pi_1 &= \frac{\partial L_{mss}}{\partial Z_1'} = 2\alpha^{-1}c_1^2 Z_1' \\
\pi_i &= \frac{\partial L_{mss}}{\partial Z_i'} = 2\alpha^{-1}c_i^2 Z_i' \\
\pi_n &= \frac{\partial L_{mss}}{\partial Z_n'} = -2\alpha^{-1}c_n^2 Z_n' \\
p_f &= \frac{\partial L_{mss}}{\partial f'} = \alpha^{-1} f' e^{-\frac{2}{3}Z_1} \\
p_\alpha &= \frac{\partial L_{mss}}{\partial \alpha'} = 0
\end{aligned}$$

Using these, we can write down the Hamiltonian:

$$H_{mss} = \frac{1}{4}\alpha\left(-c_n^{-2}\pi_n^2 + c_{n-1}^{-2}\pi_{n-1}^2 + ... + c_1^{-2}\pi_1^2 + 2e^{\frac{2}{3}Z_1}p_f^2\right) - \alpha e^{2V}R^{(10)} \tag{89}$$

Hence as before, we have the Hamiltonian constraint

$$\left(-c_n^{-2}\pi_n^2 + c_{n-1}^{-2}\pi_{n-1}^2 + ... + c_1^{-2}\pi_1^2 + 2e^{\frac{2}{3}Z_1}p_f^2\right) - K^2 e^{2Z_n} = 0. \tag{90}$$

In the gauge $\alpha = 1$, the classical equations are

$$\begin{array}{cc}
\pi_1' = -\frac{1}{3}\pi_f^2 e^{\frac{2}{3}Z_1} & Z_1' = \frac{1}{2}c_1^{-2}\pi_1 \\
\pi_n' = \frac{1}{2}K^2 e^{2Z_n} & Z_n' = -\frac{1}{2}c_n^{-2}\pi_n \\
\pi_i' = 0 & Z_i' = \frac{1}{2}c_i^{-2}\pi_i \\
p_f' = 0 & f' = e^{\frac{2}{3}Z_1}p_f
\end{array} \tag{91}$$

where $i = 2, ..., n-1$. For these values of i, we immediately write down

$$Z_i = \frac{1}{2}c_i^{-2}\pi_i\tau + Z_i(0)$$

From (56), the volume factor e^V is now

$$\begin{aligned}
e^V &= \exp\left(-c_1^2 Z_1 - ... - c_{n-1}^2 Z_{n-1} + c_n^2 Z_n\right) \\
&= A\exp\left(c_n^2 Z_n - c_1^2 Z_1\right)\exp\left[-\frac{1}{2}\left(\pi_2 + ... + \pi_{n-1}\right)\tau\right]
\end{aligned} \tag{92}$$

where $A = \exp\left[c_2^2 Z_2(0) + ... + c_{n-1}^2 Z_{n-1}(0)\right]$, so that from (50),

$$\frac{dt}{d\tau} = Ae^{c_n^2 Z_n}e^{-c_1^2 Z_1}e^{-\frac{1}{2}\pi_s\tau}. \tag{93}$$

where $\pi_s = \pi_2 + ... + \pi_{n-1}$. Thus the original time variable t can be restored in terms of τ once $Z_1(\tau)$ and $Z_n(\tau)$ are known.

Similarly as before, we can rewrite the Hamiltonian constraint (90) as

$$- c_n^{-2}\pi_n^2 + \zeta^2 + c_1^{-2}\pi_1^2 + 2e^{\frac{2}{3}Z_1}p_f^2 - K^2 e^{2Z_n} = 0. \tag{94}$$

where ζ is a constant given by

$$\zeta^2 = c_{n-1}^{-2}\pi_{n-1}^2 + ...c_2^{-2}\pi_2^2. \tag{95}$$

From the equation of motion, we have

$$\begin{aligned} 2c_1^{-2}\pi_1\frac{\partial \pi_1}{\partial Z_1} &= -\frac{4}{3}p_f^2 e^{\frac{2}{3}Z_1} \\ 2c_n^2\pi_n\frac{\partial \pi_n}{\partial Z_n} &= -2K^2e^{2Z_n} \end{aligned}$$

Integrating, we get

$$2p_f^2 e^{\frac{2}{3}Z_1} = c_1^{-2}\left(\zeta_1^2 - \pi_1^2\right) \tag{96}$$

$$K^2 e^{2Z_n} = c_n^{-2}\left(\zeta_n^2 - \pi_n^2\right) \tag{97}$$

substituting these expressions into the constraint, we get

$$\zeta^2 = c_n^{-2}\zeta_n^2 - c_1^{-2}\zeta_1^2. \tag{98}$$

Hence from equations of motion,

$$\pi_1' = -\frac{1}{6}c_1^{-2}\left(\zeta_1^2 - \pi_1^2\right) \tag{99}$$

$$\pi_n' = \frac{1}{2}c_n^{-2}\left(\zeta_n^2 - \pi_n^2\right). \tag{100}$$

Note that from (96), we must have $\zeta_1^2 \geq 0$, and because of this, from (98) ζ_n^2 also has to be non-negative. Moreover, from (96), we can induce that $|\pi_1| < \zeta_1$ and similarly, $|\pi_n| < \zeta_n$ for $K^2 > 0$, and $|\pi_n| > \zeta_n$ for $K^2 < 0$.

Consider what happens to the curvature of the 11-dimensional spacetime. From Einstein's equation, we have

$$R^{(11)} = -\frac{1}{6}e^{-2a_1X_1}\dot{f}^2 = -\frac{1}{6}e^{-2V}p_f^2 \tag{101}$$

hence the curvature blows up as the volume of the space tends to zero.

We now proceed to the quantization of the minisuperspace model. The canonical variables in the minisuperspace are now f and Z_i for $i = 1, ..., n$, and the corresponding momenta p_f and π_i for $i = 1, ..., n$. For the momentum operators, we use the following representation

$$\begin{aligned} \pi_i &= -i\frac{\partial}{\partial Z_i} \\ p_f &= -i\frac{\partial}{\partial f} \end{aligned}$$

The behaviour of the wavefunction are governed only by the constraint $\tilde{\mathcal{H}}\Psi = 0$. In our representation this gives the equation

$$\left(-c_n^{-2}\frac{\partial^2}{\partial Z_n^2} + c_{n-1}^{-2}\frac{\partial^2}{\partial Z_{n-1}^2} + ... + c_1^{-2}\frac{\partial^2}{\partial Z_1^2} + 2e^{\frac{2}{3}Z_1}\frac{\partial^2}{\partial f^2}\right)\Psi + K^2e^{2Z_n}\Psi = 0. \tag{102}$$

This equation is basically same as the equation (67) which we obtained in the previous section, but with an extra term added.

If we write

$$\Psi = e^{i\kappa_f f} e^{i\kappa_2 Z_2} \ldots e^{i\kappa_{n-1} Z_{n-1}} G(Z_1) H(Z_n)$$

then the equation becomes

$$-c_n^{-2} \frac{1}{H} \frac{\partial^2 H}{\partial Z_n^2} + c_1^{-2} \frac{1}{G} \frac{\partial^2 G}{\partial Z_1^2} - 2e^{\frac{2}{3} Z_1} \kappa_f^2 + K^2 e^{2Z_n} = \kappa^2$$

where

$$\kappa^2 = c_{n-1}^{-2} \kappa_{n-1}^2 + \ldots c_2^{-2} \kappa_2^2.$$

Separating the variables, we get the following equations for G and H:

$$\begin{aligned} \frac{\partial^2 G}{\partial Z_1^2} - \left(2c_1^2 \kappa_f^2 e^{\frac{2}{3} Z_1} - \kappa_1^2\right) G &= 0 && (103) \\ \frac{\partial^2 H}{\partial Z_n^2} - \left(c_n^2 K^2 e^{2Z_n} - \kappa_n^2\right) H &= 0 && (104) \end{aligned}$$

where

$$c_1^{-2} \kappa_n^2 - c_n^{-2} \kappa_1^2 = \kappa^2. \tag{105}$$

Note that the Z_1 equation is basically of the same form as the Z_n equation for $K^2 \geq 0$. Setting $z_1 = 3e^{\frac{1}{3} Z_1}$, we get

$$z_1^2 \frac{\partial^2 G}{\partial z_1^2} + z_1 \frac{\partial G}{\partial z_1} - \left(2c_1^2 \kappa_f^2 z_1^2 - 9\kappa_1^2\right) G = 0 \tag{106}$$

and setting $z_n = e^{Z_n}$

$$z_n^2 \frac{\partial^2 H}{\partial z_n^2} + z_n \frac{\partial H}{\partial z_n} - \left(c_n^2 K^2 z_n^2 - \kappa_n^2\right) H = 0 \tag{107}$$

therefore up to rescaling of variables, these are Bessel's equations.

The equations we got here are very similar to the equations encountered in the previous section, so we can write down the solutions straight away. The classical equation for π_1 (99) gives

$$\pi_1 = -\zeta_1 \tanh\left(\frac{1}{6} c_1^{-2} \zeta_1 \tau + \tau_0\right) \tag{108}$$

and from the equation of motion for Z_1 (91) and the constraint (96), we get the solution for Z_1:

$$Z_1 = -3 \log\left(\sqrt{2} c_1 \zeta_1^{-1} p_f \cosh\left(\frac{1}{6} c_1^{-2} \zeta_1 \tau + \tau_0\right)\right). \tag{109}$$

This is very similar to the solutions for Y_1 considered in the previous section for $K^2 > 0$. In particular, in the phase space this solution has a single branch.

The solutions for Z_n are exactly the same as the solutions for Y_1 in the previous section. Thus for $K^2 > 0$, we have

$$\pi_n = \zeta_n \tanh\left(\frac{1}{2} c_n^{-2} \zeta_n \tau + \tau_1\right) \tag{110}$$

and

$$Z_n = -\log\left(K c_n \zeta^{-1} \cosh\left(\frac{1}{2} c_n^{-2} \zeta_n \tau + \tau_1\right)\right). \tag{111}$$

Similarly, for $K^2 < 0$, we get

$$\begin{aligned}\pi_n &= \zeta_n \coth\left(\frac{1}{2}c_n^{-2}\zeta_n\tau + \tau_1\right)\\ Z_n &= -\log\left(\left|\tilde{K}c_n\zeta^{-1}\sinh\left(\frac{1}{2}c_n^{-2}\zeta_n\tau + \tau_1\right)\right|\right)\end{aligned}$$

where $\tilde{K}^2 = -K^2$. In both cases, the asymptotic behaviour as $\tau \longrightarrow \pm\infty$ is

$$\begin{aligned}Z_1 &\sim \mp\frac{1}{2}c_1^{-2}\zeta_1\tau\\ Z_n &\sim \mp\frac{1}{2}c_n^{-2}\zeta_n\tau.\end{aligned}$$

We know that the volume parameter V is given by

$$V = c_n^2 Z_n - c_1^2 Z_1 - \frac{1}{2}\pi_s\tau + const$$

so as $\tau \longrightarrow \pm\infty$,

$$V \sim \pm\frac{1}{2}\tau\left(\zeta_1 - \zeta_n \mp \pi_s\right).$$

Noting that $X_1 = \frac{1}{9}Z_1$ and $X_n = V - Z_n$, we get the asymptotic behaviour of the original variables X_1 and X_n:

$$\begin{aligned}X_1 &\sim \mp\frac{1}{3}\zeta_1\tau\\ X_n &\sim \pm\frac{1}{2}\tau\left(\zeta_1 - \left(1 - c_n^{-2}\right)\zeta_n - \pi_s\right).\end{aligned}$$

Thus as $\tau \longrightarrow \pm\infty$, $X_1 \longrightarrow -\infty$, and the qualitative behaviour of V depends on the sign of $c_\pm = \zeta_1 - \zeta_n \mp \pi_s$. Both the 4-dimensional and 11-dimensional curvatures are asymptotically proportional to e^{-2V}, so the sign of $c_\pm$ affects the behaviour of the curvature. Consider the following example. If $n = 2$, then the internal space is 7-dimensional, and moreover $\pi_s = 0$ and $\zeta_2 = c_2 c_1^{-1}\zeta_1$. This immediately gives us that $c_\pm < 0$. Hence as $\tau \longrightarrow \pm\infty$, $V \longrightarrow -\infty$.

Now look at the solutions of the Wheeler-DeWitt equation for this system (102). The solution for the Z_1 equation (106) is

$$G\left(Z_1\right) = N_1 K_{3i\kappa_1}\left(3\sqrt{2}p_f c_1 e^{\frac{1}{3}Z_1}\right) + N_2 I_{3i\kappa_1}\left(3\sqrt{2}p_f c_1 e^{\frac{1}{3}Z_1}\right).$$

Imposing the condition $|G| < \infty$, we must have $N_2 = 0$ and κ_1 real. Hence $\kappa_1^2 \geq 0$, and (105) implies that $\kappa_n^2 \geq 0$. With this condition on κ_n, the wavefunction of Z_n is exactly the same as the wavefunction of Y_1 in the previous section. Thus, the full wavefunction up to a normalization factor is given by

$$\Psi_{pos} = e^{i\kappa_f f}e^{i\kappa_2 Z_2}...e^{i\kappa_{n-1}Z_{n-1}}K_{3i\kappa_1}\left(3\sqrt{2}p_f c_1 e^{\frac{1}{3}Z_1}\right)K_{i\kappa_n}\left(Kc_n e^{Z_n}\right). \tag{112}$$

for $K^2 > 0$, and

$$\Psi_{neg} = e^{i\kappa_f f}e^{i\kappa_2 Z_2}...e^{i\kappa_{n-1}Z_{n-1}}K_{3i\kappa_1}\left(3\sqrt{2}p_f c_1 e^{\frac{1}{3}Z_1}\right)H^1_{i\kappa_n}\left(\tilde{K}c_n e^{Z_n}\right) \tag{113}$$

for $K^2 < 0$, where we have used same boundary conditions as in the case of the trivial 4-form. From the properties of $K_\nu(z)$, for $\sqrt{2}p_f c_1 e^{\frac{1}{3}Z_1} > |\kappa_1|$, both wavefunctions decay extremely fast, which ties in with the classical bound for Z_1. Similarly, Ψ_{pos} decays rapidly for $Kc_n e^{Z_n} > |\kappa_n|$.

For $Z_1, Z_n \longrightarrow -\infty$, we can decompose the wavefunction into plane waves similarly as in the case of the trivial 4-form. For Ψ_{neg} we get a non-trivial reflection probability $R_{\kappa_n} = e^{-2\kappa_n\pi}$ from the right-moving wave to the left-moving wave, which corresponds to a transition from the $\pi_n > 0$ branch to the $\pi_n < 0$ branch.

6 Concluding remarks

We have first derived the canonical formulation of the bosonic sector of eleven dimensional supergravity, together with the complete constraint algebra. The brackets of the secondary constraints vanish on the constraint surface, so all constraints are first-class and there are no new tertiary constraints. When passing to the quantum system, the constraints become conditions on the wavefunction which govern its behaviour.

By introducing particular ansätze for the metric and the 4-form we reduced the system to a minisuperspace model with a finite number of degrees of freedom. In a special case where only one spatial component has non-vanishing curvature, both the classical and quantum equations can be solved exactly. In the positive curvature case, whether with or without the 4-form, there is only one branch of the classical solution, where the universe first expands after starting out from zero size, reaches a maximum size, and then collapses again within a finite time. When the universe becomes small, the wavefunction can be written in terms of plane waves travelling in opposite directions. These waves can be interpreted as being reflections of one another, but since their coefficients are equal, the transition probability is 1. A similar scenario is considered in [11], but the effect that there is only one classical branch of the solution is achieved their by having a negative dilaton potential in the Hamiltonian, which is hard to motivate in a realistic superstring theory context.

In the negative curvature case, the classical solutions give two disconnected branches, one of which is collapsing universe, and the other branch is an expanding universe. In the context of the pre-Big Bang scenario, these can be interpreted as the pre-Big Bang and post-Big Bang branches respectively, especially since from the 4-dimensional point of view, there is a curvature singularity between the two branches. Asymptotic decomposition of the wavefunction into plane waves this time yields a non-trivial transition probability between the branches.

So as we have seen, the curvature term and the 4-form term in our minisuperspace models, in terms of determining the behaviour of the solution, plays the same role as the dilaton potential in the gravi-dilaton systems derived from string theory. This is quite remarkable because these terms naturally from the supergravity action, whereas the dilaton potentials are put in by hand. It would be interesting to investigate what happens when there is more than one spatial curvature term. Such an ansatz would be a generalization of the Freund-Rubin solution of M-theory [26], where the eleven-dimensional space is of the form $AdS_4 \times S^7$ with particular scale factors for each component. In particular, if the 3-space is curved, then there could possibly be more interaction 4-form term and the curvature term. Also, in further work, a less restrictive metric ansatz with a non-trivial moduli space could be studied, to see how the moduli space parameters evolve and what is the behaviour of their wavefunctions. In particular, it would be interesting to study compactifications on manifolds of special holonomy with time-dependent moduli. This could either involve compactifications on general G_2-holonomy manifolds or maybe on a Calabi-Yau space times a cirle. In the latter case, it could be investigated how mirror symmetry [27] is manifested from the point of view of a minisuperspace quantization.

Study of M-theory minisuperspace models seems to be a promising area where there is still much left to be uncovered, and which will hopefully aid us in the quest to further understand M-theory.

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A Appendix

In this Appendix we give the details of the calculations involved in deriving the expression (41) for the Poisson bracket $\left[\tilde{\mathcal{H}},\tilde{\mathcal{H}}\right]$. To do this, we first need to know the bracket $\left[\tilde{\pi}^{abc},\tilde{\pi}'^{efg}\right]$. Expanding, we have

$$\begin{aligned}
\left[\tilde{\pi}^{abc},\tilde{\pi}'^{efg}\right] &= \left[\pi^{abc}+\frac{32}{12^4}\eta^{abcd_1\ldots d_7}A_{d_1d_2d_3}\partial_{[d_4}A_{d_5d_6d_7]},\pi'^{efg}+\frac{32}{12^4}\eta^{efga_1\ldots a_7}A'_{a_1a_2a_3}\partial'_{[a_4}A'_{a_5a_6a_7]}\right]\\
&= \left[\pi^{abc},\frac{32}{12^4}\eta^{efga_1\ldots a_7}A'_{a_1a_2a_3}\partial'_{[a_4}A'_{a_5a_6a_7]}\right]+\left[\frac{32}{12^4}\eta^{abcd_1\ldots d_7}A_{d_1d_2d_3}\partial_{[d_4}A_{d_5d_6d_7]},\pi'^{efg}\right]\\
&= -\frac{32}{12^4}\eta^{efga_1\ldots a_7}\left(A'_{a_1a_2a_3}\delta^{abc}{}_{a_5a_6a_7}\delta_{,a'_4}\left(x,x'\right)+\delta^{abc}{}_{a_1a_2a_3}\delta\left(x,x'\right)\partial'_{[a_4}A'_{a_5a_6a_7]}\right)\\
&\quad+\frac{32}{12^4}\eta^{abcd_1\ldots d_7}\left(A_{d_1d_2d_3}\delta^{efg}{}_{d_5d_6d_7}\delta_{,d_4}\left(x,x'\right)+\delta^{efg}{}_{d_1d_2d_3}\delta\left(x,x'\right)\partial_{[d_4}A_{d_5d_6d_7]}\right)\\
&= \frac{32}{12^4}\eta^{abcefga_1\ldots a_4}\Big(A'_{a_1a_2a_3}\delta_{,a'_4}\left(x,x'\right)+\delta\left(x,x'\right)\partial'_{[a_1}A'_{a_2a_3a_4]}\\
&\qquad+A_{a_1a_2a_3}\delta_{,a_4}\left(x,x'\right)+\delta\left(x,x'\right)\partial_{[a_1}A_{a_2a_3a_4]}\Big)\\
&= \frac{32}{12^4}\eta^{abcefga_1\ldots a_4}\left(A'_{a_1a_2a_3}\delta_{,a'_4}\left(x,x'\right)+A_{a_1a_2a_3}\delta_{,a_4}\left(x,x'\right)+2\delta\left(x,x'\right)\partial_{[a_1}A_{a_2a_3a_4]}\right)
\end{aligned}\tag{114}$$

So, for the $\left[\tilde{\mathcal{H}},\tilde{\mathcal{H}}'\right]$ bracket, we have:

$$\begin{aligned}
\left[\tilde{\mathcal{H}},\tilde{\mathcal{H}}'\right] &= \left[\mathcal{H}+\frac{1}{48}\gamma^{\frac12}F_{abcd}F^{abcd}+3\gamma^{-\frac12}\tilde{\pi}^{abc}\tilde{\pi}_{abc},\mathcal{H}'+\frac{1}{48}\gamma'^{\frac12}F'_{abcd}F'^{abcd}+3\gamma'^{-\frac12}\tilde{\pi}'^{abc}\tilde{\pi}'_{abc}\right]\\
&= \left[\mathcal{H},\mathcal{H}'\right]+\gamma^{\frac12}\gamma'^{-\frac12}F^{abcd}\tilde{\pi}'_{efg}\left[A_{bcd},\tilde{\pi}'^{efg}\right]_{,a}-\gamma'^{\frac12}\gamma^{-\frac12}F'^{abcd}\tilde{\pi}_{efg}\left[A'_{bcd},\tilde{\pi}^{efg}\right]_{,a'}\\
&\qquad+36\gamma^{-\frac12}\gamma'^{-\frac12}\tilde{\pi}_{abc}\tilde{\pi}'_{efg}\left[\tilde{\pi}^{abc},\tilde{\pi}'^{efg}\right]\\
&= \left[\mathcal{H},\mathcal{H}'\right]+\gamma^{\frac12}\gamma'^{-\frac12}F^{aefg}\tilde{\pi}'_{efg}\delta_{,a}\left(x,x'\right)-\gamma'^{\frac12}\gamma^{-\frac12}F'^{aefg}\tilde{\pi}_{efg}\delta_{,a'}\left(x,x'\right)\\
&\quad+\frac{8}{12^2}\gamma^{-\frac12}\gamma'^{-\frac12}\tilde{\pi}_{abc}\tilde{\pi}'_{efg}\eta^{abcefga_1a_2a_3a_4}\left(A'_{a_1a_2a_3}\delta_{,a'_4}\left(x,x'\right)+A_{a_1a_2a_3}\delta_{,a_4}\left(x,x'\right)\right)
\end{aligned}\tag{115}$$

In the first line cross terms involving $\mathcal{H}$ vanish because the form terms involve no derivatives of γ_{ab} and $\mathcal{H}$ does not involve any derivatives of π^{ab}. Note that the undifferentiated δ-function term in $\tilde{\pi}_{abc}\tilde{\pi}'_{efg}\left[\tilde{\pi}^{abc},\tilde{\pi}'^{efg}\right]$ vanishes, because $\eta^{abcefg}\tilde{\pi}_{abc}\tilde{\pi}_{efg}=0$. Let ξ_1 and ξ_2 be arbitrary test functions. Then

$$\begin{aligned}
\int\!\!\int\left[\tilde{\mathcal{H}},\tilde{\mathcal{H}}'\right]\xi_1\xi'_2dxdx' &= \int\!\!\int\left[\mathcal{H},\mathcal{H}'\right]\xi_1\xi'_2dxdx'
\end{aligned}\tag{116}$$

$$+\int\!\!\int\left(\left(\gamma\gamma'^{-1}\right)^{\frac12}F^{a_1\ldots a_4}\tilde{\pi}'_{a_2a_3a_4}\delta_{,a_1}\left(x,x'\right)-\left(\gamma'\gamma^{-1}\right)^{\frac12}F'^{a_1\ldots a_4}\tilde{\pi}_{a_2a_3a_4}\delta_{,a'_1}\left(x,x'\right)\right)\xi_1\xi'_2dxdx'$$

$$+\frac{8}{12^2}\int\!\!\int\left(\gamma\gamma'\right)^{-\frac12}\tilde{\pi}_{abc}\tilde{\pi}'_{efg}\eta^{abcefga_1\ldots a_4}\left(A'_{a_1a_2a_3}\delta_{,a'_4}\left(x,x'\right)+A_{a_1a_2a_3}\delta_{,a_4}\left(x,x'\right)\right)\xi_1\xi'_2dxdx'$$

We know from [3] that

$$\int\!\!\int\left[\mathcal{H},\mathcal{H}'\right]\xi_1\xi'_2dxdx'=\int\chi^a\left(\xi_1\xi_{2,a}-\xi_{1,a}\xi_2\right)dx\tag{117}$$

The second line in (116) becomes

$$\begin{aligned}
&\int\int\left(\gamma^{\frac{1}{2}}\gamma'^{-\frac{1}{2}}F^{aefg}\tilde{\pi}'_{efg}\delta_{,a}\left(x,x'\right)-\gamma'^{\frac{1}{2}}\gamma^{-\frac{1}{2}}F'^{aefg}\tilde{\pi}_{efg}\delta_{,a'}\left(x,x'\right)\right)\xi_1\xi_2'dxdx'\\
=&-\int\int\left(\gamma'^{-\frac{1}{2}}\tilde{\pi}'_{efg}\left(\gamma^{\frac{1}{2}}F^{aefg}\xi_1\right)_{,a}\xi_2'-\gamma^{-\frac{1}{2}}\tilde{\pi}_{efg}\left(\gamma'^{\frac{1}{2}}F'^{aefg}\xi_2'\right)_{,a'}\xi_1\right)\delta\left(x,x'\right)dxdx'\\
=&-\int\left(\gamma^{-\frac{1}{2}}\tilde{\pi}_{efg}\left(\gamma^{\frac{1}{2}}F^{aefg}\xi_1\right)_{,a}\xi_2-\gamma^{-\frac{1}{2}}\tilde{\pi}_{efg}\left(\gamma^{\frac{1}{2}}F^{aefg}\xi_2\right)_{,a}\xi_1\right)dx\\
=&\int F^{a}{}_{efg}\tilde{\pi}^{efg}\left(\xi_1\xi_{2,a}-\xi_{1,a}\xi_2\right)dx
\end{aligned}$$

Now look at the third line in (116). After integrating by parts, and integrating out the δ-function we get

$$\int\frac{8}{12^2}\eta^{abcefga_1a_2a_3a_4}\gamma^{-1}\tilde{\pi}_{abc}\tilde{\pi}_{efg}A_{a_1a_2a_3}\left(\xi_1\xi_{2,a_4}-\xi_{1,a_4}\xi_2\right)dx=0$$

again because $\eta^{abcefg}\tilde{\pi}_{abc}\tilde{\pi}_{efg}=0$.

Thus

$$\begin{aligned}
\int\int\left[\tilde{\mathcal{H}},\tilde{\mathcal{H}}'\right]\xi_1\xi_2'dxdx' &= \int\left(\chi^a+F^{a}{}_{efg}\tilde{\pi}^{efg}\right)\left(\xi_1\xi_{2,a}-\xi_{1,a}\xi_2\right)dx\\
&= \int\tilde{\chi}^a\left(\xi_1\xi_{2,a}-\xi_{1,a}\xi_2\right)dx
\end{aligned}\tag{118}$$

Correspondingly,

$$\left[\tilde{\mathcal{H}},\tilde{\mathcal{H}}'\right]=2\tilde{\chi}^a\delta_{,a}\left(x,x'\right)+\tilde{\chi}^a{}_{,a}\delta\left(x,x'\right)$$

which is completely analogous to the untilded expression. In particular, $\left[\tilde{\mathcal{H}},\tilde{\mathcal{H}}'\right]$ vanishes on the constraint surface.

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